CONSTRAINED DYNAMICAL SYSTEMS: SEPARATION OF CONSTRAINTS INTO FIRST AND SECOND CLASSES

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Abstract

In the Dirac approach to the generalized Hamiltonian formalism, dynamical systems with first- and second-class constraints are investigated. The classification and separation of constraints into the first- and second-class ones are presented with the help of passing to an equivalent canonical set of constraints. The general structure of second-class constraints is clarified.

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1   Introduction

The generalized Hamiltonian formalism is a classical basis of the gauge theories [1]. Originally the theories with the only first-class constraints have played a main role among these theories, because the gauge degrees of freedom are stipulated by the mentioned constraints. However, for example, theories with the massive vector fields, supersymmetric and superstring models introduce in consideration also the constraints of second class. But the general case the generalized Hamiltonian dynamics, when in addition to first-class constraints the second-class constraints are present also in a theory, is studied relatively slightly up to now. Actually, there are two approaches to treating constrained systems. In one approach, using the classification of constraints into the first and second class, the second-class constraints are disposed of by the Dirac brackets method [1] - [4]. Here the separation of constraints into first- and second-class ones is needed, a possibility of which was indicated by Dirac. But only in recent years there have appeared the real methods for such separation [5] - [7], developed, however, in the framework of the modified generalized Hamiltonian formalism. In these papers, other than Dirac schemes were used for the constraint proliferation, therefore, the question arises naturally about equivalence of the constraint sets obtained in these works to the Dirac set. Generally in the investigations of the dynamical systems with second-class constraints there is a tendency (may be, not always justified) to modify initial formulation of the generalized Hamiltonian dynamics [5] - [9]. Note that there is another more recent approach [10] which does not apply the above classification of constraints and where one has shown that for some Lagrangian systems the basic bracket relations can be obtained without using the usual Dirac brackets.

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In this paper, we shall follow the conventional Dirac approach. Assuming a complete set of constraints to be obtained according to the Dirac scheme for breeding the constraints, we shall show that we can separate them into the first- and second-class ones without modifying this scheme and solve the problem of passing to an equivalent canonical set of constraints to be used in a subsequent paper for deriving the local-symmetry transformation generator and for proving the fact that second-class constraints do not contribute to the law of these transformations unlike the assertions appeared recently in the literature [11] - [13].

2   Classification and Separation of First- and Second-Class Constraints

Here we restrict for simplicity ourselves to a system with a finite number of the degrees of freedom N described by a degenerate Lagrangian $L(q, \dot{q})$, where $q = (q_1, \cdots, q_N)$ and $\dot{q} = dq/dt = (\dot{q}_1, \cdots, \dot{q}_N)$ are generalized coordinates and velocities, respectively (all subsequent considerations may be extended to the field theory by a standard way). After passing to the Hamiltonian formalism, let A primary constraints be obtained in the phase space (q, p). Further, provided that equations of motion are self-consistent, all constraints are established according to the Dirac procedure of breeding the constraints. The primary constraints will be denoted by ϕ_α^1, where $\alpha = 1, \cdots, A$ and

$$N - A = \text{rank} \left\| \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \right\|. \tag{1}$$

According to the Dirac scheme, from the stationary-state condition of the primary constraints $\dot{\phi}_\alpha^1 = \{\phi_\alpha^1, H_T\} = 0$ (H_T is the total Hamiltonian) we obtain the secondary constraints, denoted by ϕ_α^2, from the stationary-state condition of which the tertiary constraints ϕ_α^3 are obtained, etc. For consistent theories this procedure is finished after the definite number of steps $M_\alpha - 1$, i.e. $\dot{\phi}_\alpha^{M_\alpha} = 0$ with taking account of all the previous constraints. So, we have a system of constraints $\phi_\alpha^{m_\alpha}$, where $\alpha = 1, \cdots, A$ and $m_\alpha = 1, 2, \cdots, M_\alpha$ ($\sum_{\alpha=1}^A M_\alpha = M$). The set of constraints $\phi_\alpha^{m_\alpha}$ is complete and irreducible [1]. Furthermore, let

$$\text{rank} \left\| \{\phi_\alpha^{m_\alpha}, \phi_\beta^{m_\beta}\} \right\| = 2R < M, \tag{2}$$

which implies the presence of $2R$ constraints of second class $\Psi_a^{m_a}$ and $M - 2R$ constraints of first class $\Phi_\alpha^{m_\alpha}$ subjected to the relation:

$$\{\Phi_\alpha^{m_\alpha}, \Phi_\beta^{m_\beta}\} \stackrel{\Sigma}{=} \{\Phi_\alpha^{m_\alpha}, \Psi_a^{m_a}\} \stackrel{\Sigma}{=} 0, \tag{3}$$

$$\{\Psi_a^{m_a}, \Psi_b^{m_b}\} \stackrel{\Sigma}{=} F_{a\ b}^{m_a m_b} \neq 0 \tag{4}$$

($\stackrel{\Sigma}{=}$ means this equality to hold on the surface of all constraints Σ). The constraint sets (Φ, Ψ) and $\phi_\alpha^{m_\alpha}$ are related with each other by the equivalence transformation:

$$\begin{pmatrix} \mathbf{\Phi} \\ \mathbf{\Psi} \end{pmatrix} = \mathbf{S} \begin{pmatrix} \phi \end{pmatrix}, \qquad \det \mathbf{S} \stackrel{\Sigma}{\neq} 0.$$

A possibility of constructing the set (Φ, Ψ) was indicated by Dirac. However, for practical aims (for example, to elucidate a role of second-class constraints in gauge transformations [14]) the explicit form of set (Φ, Ψ) is to be known. In what follows we shall busy ourself to obtain this set of constraints through several successive stages being remain in the framework of the Dirac approach and having in mind its employment next for deriving the local-symmetry transformation generator.

Let us consider the antisymmetric matrix $\mathbf{K}^{11}$ with elements $K^{11}_{\alpha\beta} = \{\phi^1_\alpha, \phi^1_\beta\}$, and let

$$\operatorname{rank}\left\|K^{11}_{\alpha\beta}\right\| \stackrel{\Sigma_1}{=} A_1 = 2R_1 < A \tag{5}$$

(Σ_1 is a primary constraint surface), i.e. A_1 primary constraints exhibit their nature of second class already at this stage (more exactly, they are candidates for this role provided that we shall be able to develop the following procedure). One can regard the principal minor of rank A_1, disposed in the left upper corner of the matrix $\mathbf{K}^{11}$, to be not equal to zero. Write down

$$\{\phi^1_\alpha, \phi^1_\beta\} = f_{\alpha\beta\gamma}\, \phi^1_\gamma + D_{\alpha\beta}, \quad \alpha, \beta, \gamma = 1, \cdots, A \tag{6}$$

where

$$D_{\alpha\beta} \stackrel{\Sigma_1}{=} F_{\alpha\beta}.$$

Among $F_{1\alpha}$ ($\alpha = 2, \cdots, A_1$) at least one element is non-zero in accordance with the supposition (5). Renumbering the constraints one can always obtain that $F_{12} \neq 0$.

Pass to a new set of constraints:

$$
\begin{aligned}
{}^1\phi^1_1 &= \phi^1_1\,, & {}^1\phi^1_2 &= \phi^1_2\,, \\
{}^1\phi^1_\alpha &= \phi^1_\alpha + {}^1u_{\alpha 1}\, \phi^1_1 + {}^1u_{\alpha 2}\, \phi^1_2\,, & \alpha &= 3, \cdots, A.
\end{aligned} \tag{7}
$$

The left superscripts indicate a stage of our procedure and will be omitted in the resultant expressions. Determine the coefficients ${}^1u_{\alpha 1}$ and ${}^1u_{\alpha 2}$, which are functions of q and p, by the following expressions:

$$
{}^1u_{\alpha 1} = \frac{D_{2\alpha}}{D_{12}}\,, \qquad {}^1u_{\alpha 2} = -\frac{D_{1\alpha}}{D_{12}}\,, \qquad \alpha = 3, 4, \cdots, A, \tag{8}
$$

to guarantee the fulfilment of requirements:

$$
\begin{aligned}
\{{}^1\phi^1_1\,, {}^1\phi^1_\alpha\} &= f_{1\alpha\beta}\, \phi^1_\beta + D_{1\alpha} + \{\phi^1_1\,, {}^1u_{\alpha 1}\}\, \phi^1_1 + f_{12\beta}\, {}^1u_{\alpha 2}\, \phi^1_\beta \\
&\quad + D_{12}\, {}^1u_{\alpha 2} + \{\phi^1_1\,, {}^1u_{\alpha 2}\}\, \phi^1_2 \stackrel{\Sigma_1}{=} 0, \\
\{{}^1\phi^1_2\,, {}^1\phi^1_\alpha\} &= f_{2\alpha\beta}\, \phi^1_\beta + D_{2\alpha} + \{\phi^1_2\,, {}^1u_{\alpha 1}\}\, \phi^1_1 + f_{21\beta}\, {}^1u_{\alpha 1}\, \phi^1_\beta \\
&\quad + D_{21}\, {}^1u_{\alpha 1} + \{\phi^1_2\,, {}^1u_{\alpha 2}\}\, \phi^1_2 \stackrel{\Sigma_1}{=} 0.
\end{aligned} \tag{9}
$$

With the help of (6) and (8), it is easily seen that

$$
{}^1D_{12} = -{}^1D_{21} \stackrel{\Sigma_1}{=} {}^1F_{12} = F_{12} \neq 0\,, \qquad {}^1D_{\alpha\beta} \stackrel{\Sigma_1}{=} F_{\alpha\beta} = 0, \qquad \alpha = 1, 2, \quad \beta = 3, 4, \cdots, A.
$$

So, by means of the transformation

$$
{}^1\phi^1_\alpha = {}^1\Lambda_{\alpha\beta}\, \phi^1_\beta, \qquad \det\|{}^1\Lambda_{\alpha\beta}\| = 1, \tag{10}
$$

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$$
{}^1\mathbf{\Lambda} \;=\; \begin{pmatrix} {}^1\mathbf{I}_1 & {}^1\mathbf{O} \\ {}^1\mathbf{U} & {}^1\mathbf{I}_2 \end{pmatrix}, \tag{11}
$$

where ${}^1\mathbf{I}_1$, ${}^1\mathbf{I}_2$ and ${}^1\mathbf{O}$ are the unit 2×2-, $(A-2) \times (A-2)$- and zero $2 \times (A-2)$-blocks, respectively, and

$$
{}^1\mathbf{U} = \begin{pmatrix} {}^1u_{31} & {}^1u_{32} \\ \vdots & \vdots \\ {}^1u_{A1} & {}^1u_{A2} \end{pmatrix},
$$

we obtain at the first stage of our procedure:

$$
{}^1K^{11}_{\alpha\beta} \;=\; \{ {}^1\phi^1_\alpha \,,\, {}^1\phi^1_\beta \} \;=\; {}^1\Lambda_{\alpha\sigma}\, {}^1\Lambda_{\beta\tau}\, K^{11}_{\sigma\tau} \;+\; O(\phi^1_\alpha), \tag{12}
$$

$$
{}^1\mathbf{K}^{11} \;\overset{\Sigma_1}{=}\; \begin{pmatrix} F_{12} \cdot \mathbf{J} & \mathbf{O} \\ \mathbf{O} & \| {}^1\mathcal{F}_{\alpha\beta} \|\,(\alpha, \beta = 3, 4, \cdots, A) \end{pmatrix},
$$

where $\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\mathbf{O}$ are zero blocks, and $\| {}^1\mathcal{F}_{\alpha\beta} \|$ is $(A-2) \times (A-2)$-block, which must be reduced to the quasidiagonal form at the next stages of procedure.

It is evident, we have

$$
\mathrm{rank}\, \left\| {}^1\mathcal{F}_{\alpha\beta} \right\|_{\alpha,\beta=3,4,\cdots,A} = A_1 - 2,
$$

therefore among elements ${}^1F_{3\beta}$ $(\beta = 4, \cdots, A_1)$ at least one (let ${}^1F_{34}$) is not equal to zero. Repeating the above procedure in respect to this block, i.e. making the transformation

$$
{}^2\phi^1_\alpha = \begin{cases} {}^1\phi^1_\alpha\,, & \alpha = 1, 2, 3, 4, \\ {}^1\phi^1_\alpha \;+\; {}^2u_{\alpha 3}\, {}^1\phi^1_3 \;+\; {}^2u_{\alpha 4}\, {}^1\phi^1_4\,, & \alpha = 5, 6, \cdots, A, \end{cases} \tag{13}
$$

where functions ${}^2u_{\alpha 3}$ and ${}^2u_{\alpha 4}$ are determined as

$$
{}^2u_{\alpha 3} \;=\; \frac{{}^1D_{4\alpha}}{{}^1D_{34}}, \qquad {}^2u_{\alpha 4} \;=\; -\frac{{}^1D_{3\alpha}}{{}^1D_{34}}, \qquad \alpha = 5, 6, \cdots, A, \tag{14}
$$

we satisfy the requirements

$$
\{ {}^2\phi^1_3 \,,\, {}^2\phi^1_\alpha \} \overset{\Sigma_1}{=} 0, \qquad \{ {}^2\phi^1_4 \,,\, {}^2\phi^1_\alpha \} \overset{\Sigma_1}{=} 0. \tag{15}
$$

One can see with the help of (6), (14) and (15) that

$$
{}^2D_{34} \;=\; -{}^2D_{43} \overset{\Sigma_1}{=} {}^1F_{34} \neq 0, \quad {}^2D_{\alpha\beta} \overset{\Sigma_1}{=} {}^2F_{\alpha\beta} = 0, \quad \alpha = 3, 4, \;\; \beta = 5, 6, \cdots, A
$$

and, furthermore,

$$
{}^2D_{1\beta} \;=\; {}^1D_{1\beta} \;\; (\beta = 2, 3, 4), \quad {}^2D_{23} \;=\; {}^1D_{23}\,, \quad {}^2D_{24} \;=\; {}^1D_{24}\,,
$$

$$
{}^2D_{\alpha\beta} \overset{\Sigma_1}{=} {}^1D_{\alpha\beta} \;\; (\alpha = 1, 2, \;\; \beta = 5, 6, \cdots, A),
$$

i.e. the structure of zero blocks and the principal left minor, which is obtained at the first stage, has survived.

So, at the second stage, with the help of the transformation

$$^2\phi^1_\alpha = {}^2\Lambda_{\alpha\beta}\,{}^1\phi^1_\beta = {}^2\Lambda_{\alpha\beta}\,{}^1\Lambda_{\beta\sigma}\,\phi^1_\sigma\,, \qquad \det\|{}^2\Lambda_{\alpha\beta}\,{}^1\Lambda_{\beta\sigma}\| = 1, \tag{16}$$

$$^2\Lambda = \begin{pmatrix} {}^2\mathbf{I}_1 & {}^2\mathbf{O} \\ {}^2\mathbf{U} & {}^2\mathbf{I}_2 \end{pmatrix}, \tag{17}$$

where $^2\mathbf{I}_1$, $^2\mathbf{I}_2$ and $^2\mathbf{O}$ are the unit 4×4-, $(A-4)\times(A-4)$- and zero $4\times(A-4)$-matrices, respectively, and

$$^2\mathbf{U} = \begin{pmatrix} 0 & 0 & {}^2u_{53} & {}^2u_{54} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & {}^2u_{A3} & {}^2u_{A4} \end{pmatrix},$$

we obtain

$$^2\mathbf{K}^{11} \overset{\Sigma_1}{=} \left(\begin{array}{cc|c} \begin{matrix} F_{12}\cdot\mathbf{J} & \mathbf{O} \\ \mathbf{O} & {}^1F_{34}\cdot\mathbf{J} \end{matrix} & & \mathbf{O} \\ \hline & \mathbf{O} & \|{}^2\mathcal{F}_{\alpha\beta}\|(\alpha,\beta=5,6,\cdots,A) \end{array} \right). \tag{18}$$

Iterating now this procedure $R_1 = A_1/2$ times, we shall receive the matrix $^{R_1}\mathbf{K}^{11} = \left\|\{\,{}^{R_1}\phi^1_\alpha\,,\,{}^{R_1}\phi^1_\beta\,\}\right\|$ in the quasidiagonal form on the primary-constraint surface Σ_1:

$$^{R_1}\mathbf{K}^{11} \overset{\Sigma_1}{=} \left(\begin{array}{cccc|c} F_{12}\cdot\mathbf{J} & \mathbf{O} & \dots & \mathbf{O} & \\ \mathbf{O} & {}^1F_{34}\cdot\mathbf{J} & \dots & \mathbf{O} & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots & \\ \mathbf{O} & \mathbf{O} & \dots & {}^{R_1-1}F_{A_1-1\ A_1}\cdot\mathbf{J} & \\ \hline & & \mathbf{O} & & \mathbf{O} \end{array} \right). \tag{19}$$

The corresponding equivalent set of primary constraints is determined by the relation:

$$^{R_1}\phi^1_\alpha = {}^{R_1}\Lambda_{\alpha\beta}\,{}^{R_1-1}\Lambda_{\beta\gamma}\cdots{}^1\Lambda_{\sigma\tau}\,{}^1\phi^1_\tau = \overline{\Lambda}_{\alpha\beta}\,\phi^1_\beta\,, \qquad \det\overline{\Lambda} = 1. \tag{20}$$

Among A primary constraints, $A_1 = 2R_1$ functions had exhibited their nature of second class already in interaction with each other. In described procedure it is important that every subsequent stage preserves the structure of zero blocks and principal left minor obtained at the preceding stage. We shall denote the second-class constraints by the letter $\psi(\Psi)$. Thus the following set of primary constraints is obtained:

$$[\psi^1_{a_1}]^{A_1}_{a_1=1}\,, \qquad [\phi^1_{\alpha_1}]^{A-A_1}_{\alpha_1=1} \tag{21}$$

with properties

$$\{\psi^1_{a_1},\psi^1_{b_1}\} \overset{\Sigma_1}{=} \begin{cases} F_{a_1 b_1} \neq 0, & a_1 = 2k+1,\ b_1 = 2k+2\ \text{ and} \\ & \text{conversely } (k=0,1,\cdots,A_1-2), \\ 0, & \text{in other cases,} \end{cases} \tag{22}$$

$$\{\psi^1_{a_1},\phi^1_{\alpha_1}\} \overset{\Sigma_1}{=} 0, \qquad \{\phi^1_{\alpha_1},\phi^1_{\beta_1}\} \overset{\Sigma_1}{=} 0. \tag{23}$$

It is clear that constraints $\psi^1_{a_1}$ do not generate secondary constraints. Furthermore, one can attain that

$$\{\psi^1_{a_1},\phi^{m_\alpha}_\alpha\} \overset{\Sigma_1}{=} 0, \qquad m_\alpha = 2,\cdots,M_\alpha. \tag{24}$$

To this end, we shall make the transformation

$$^1\phi_\alpha^{m_\alpha} = \phi_\alpha^{m_\alpha} + C_{\alpha b_1}^{m_\alpha} \psi_{b_1}^1 . \tag{25}$$

Then using the definition

$$\{\psi_{a_1}^1, \phi_\alpha^{m_\alpha}\} \overset{\Sigma_1}{\triangleq} F_{a_1\ \alpha}^{1\ m_\alpha}$$

and taking account of (22), we shall meet the request (24) provided that

$$C_{\alpha b_1}^{m_\alpha} = -\frac{F_{a_1\ \alpha}^{1\ m_\alpha}}{F_{a_1 b_1}} ,$$

where if $a_1 = 2k + 1$, then $b_1 = 2k + 2$ and conversely $(k = 0, 1, \cdots, A_1 - 2)$.

Now let us turn to $\phi_{\alpha_1}^{m_{\alpha_1}}$, $\alpha_1 = 1, \cdots, A - A_1$. Let

$$\text{rank} \left\| \{\phi_{\alpha_1}^1, \phi_{\beta_1}^2\} \right\| \overset{\Sigma}{\triangleq} A_2 < A - A_1. \tag{26}$$

Furthermore, we have

$$\{\phi_{\alpha_1}^1, \phi_{\beta_1}^2\} \overset{\Sigma}{=} \{\phi_{\beta_1}^1, \phi_{\alpha_1}^2\}. \tag{27}$$

In considering the matrix $\left\| \{\phi_{\alpha_1}^1, \phi_{\beta_1}^2\} \right\|$ one can regard the principal minor of rank A_2, disposed in the left upper corner of this matrix, to be not equal to zero. We denote it by

$$\mathbf{K}^{12} = \left\| \{\phi_{a_2}^1, \phi_{b_2}^2\} \right\| , \quad \text{where} \quad a_2, b_2 = 1, \cdots, A_2.$$

Using the procedure which is analogous to the one for quasidiagonalization of the matrix $\mathbf{K}^{11}$, we shall obtain the matrix $\mathbf{K}^{12}|_\Sigma$ in the diagonal form. To this end, we notice at first that $\{\phi_1^1, \phi_1^2\} \overset{\Sigma}{\neq} 0$. We make the transformation

$$^1\phi_1^1 = \phi_1^1 , \quad ^1\phi_a^1 = \phi_a^1 + {}^1u_{a1} \phi_1^1 , \quad a = 2, \cdots, A_2, \tag{28}$$

from here

$$^1\phi_1^2 = \phi_1^2 , \quad ^1\phi_a^2 = \phi_a^2 + {}^1u_{a1} \phi_1^2 , \quad a = 2, \cdots, A_2, \tag{29}$$

where $^1u_{a1}$ is taken as

$$^1u_{a1} = -\frac{D_{1a}^{12}}{D_{11}^{12}} \tag{30}$$

to satisfy the requirement

$$\{^1\phi_1^1 , {}^1\phi_a^2\} \overset{\Sigma}{=} \{^1\phi_a^1 , {}^1\phi_1^2\} \overset{\Sigma}{=} 0.$$

Moreover, we have

$$\{^1\phi_2^1 , {}^1\phi_2^2\} \overset{\Sigma}{=} {}^1F_{22}^{12} = F_{22}^{12} - \frac{(F_{12}^{12})^2}{F_{11}^{12}} \neq 0. \tag{31}$$

Further making the transformation

$$^2\phi_2^1 = {}^1\phi_2^1 , \quad ^2\phi_a^1 = {}^1\phi_a^1 + {}^2u_{a2} {}^1\phi_2^1 , \quad a = 3, \cdots, A_2, \tag{32}$$

and, therefore,

$$^2\phi_2^2 = {}^1\phi_2^2 , \quad ^2\phi_a^2 = {}^1\phi_a^2 + {}^2u_{a2} {}^1\phi_2^2 , \quad a = 3, \cdots, A_2, \tag{33}$$

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we determine $^2u_{a2}$ as

$$^2u_{a2} = -\frac{{}^1D_{2a}^{12}}{{}^1D_{12}^{22}} \tag{34}$$

to satisfy the requirement

$$\{{}^2\phi_2^1 \,,\, {}^2\phi_a^2\} \overset{\Sigma}{=} \{{}^2\phi_a^2 \,,\, {}^2\phi_2^2\} \overset{\Sigma}{=} 0.$$

Furthermore, we have

$$\{{}^2\phi_3^1 \,,\, {}^2\phi_3^2\} \overset{\Sigma}{=} {}^2F_{33}^{12} = {}^1F_{33}^{12} - \frac{({}^1F_{23}^{12})^2}{{}^1F_{22}^{12}} \neq 0. \tag{35}$$

So, we have obtained

$$^2\mathbf{K}^{12} \overset{\Sigma}{=} \left(\begin{array}{cc|c} F_{11}^{12} & 0 & \\ 0 & {}^1F_{22}^{12} & \mathbf{O} \\ \hline & \mathbf{O} & \|{}^2\mathcal{F}_{ab}\|(a,b=3,4,\cdots,A_2) \end{array} \right).$$

Continuing this process, we shall deduce at the $(A_2 - 1)$ stage that with the help of the equivalence transformation of those primary constraints which have the nonvanishing Poisson brackets on Σ with their secondary constraints (the latter are obtained by the Dirac procedure), the matrix $\mathbf{K}^{12}|_\Sigma$ is leaded to the diagonal form with the nonvanishing diagonal elements

$$\{\psi_{a_2}^1, \psi_{a_2}^2\} \overset{\Sigma}{=} F_{a_2}^1{}^2{}_{a_2} \neq 0 \quad (a_2 = 1,\cdots,A_2).$$

Note, the constraints, which have exhibited their nature of second class, are denoted again by the letter ψ.

Here we notice that sometimes it may be useful to change the matrix $\mathbf{K}^{12}|_\Sigma$ into the unit form. For this we make transformation

$$^1\psi_{a_2}^1 = C_{a_2b_2}\psi_{b_2}^1 \,, \tag{36}$$

then

$$^1\dot{\psi}_{a_2}^1 \overset{\Sigma_1}{=} C_{a_2b_2}\psi_{b_2}^2 \equiv {}^1\psi_{a_2}^2 \,. \tag{37}$$

Coefficients $C_{a_2b_2}$ will be evaluated from the requirement

$$\{{}^1\psi_{a_2}^1 \,,\, {}^1\psi_{b_2}^2\} \overset{\Sigma}{=} \delta_{a_2b_2}.$$

We get the transformation matrix:

$$\mathbf{C} = (\mathbf{F}^{12})^{-1/2}. \tag{38}$$

So, it is clear that constraints $\psi_{a_2}^2$ do not generate the tertiary ones. With taking account of the definition of constraints and the properties of the Poisson brackets, the property (27) gives rise to

$$\{\psi_{a_2}^2, \psi_{b_2}^2\} \overset{\Sigma}{=} 0. \tag{39}$$

Furthermore, with the help of transformation

$$^1\phi_{\alpha_2}^{m_{\alpha_2}} = \phi_{\alpha_2}^{m_{\alpha_2}} + C_{\alpha_2}^{m_{\alpha_2}m_{a_2}}{}_{a_2} \psi_{a_2}^{m_{a_2}} \tag{40}$$

one can ensure a realization of the following equality:

$$\{\psi_{a_2}^{m_{a_2}}, \phi_{\alpha_2}^{m_{\alpha_2}}\} \overset{\Sigma}{=} 0, \qquad m_{a_2} = 1, 2, \quad \alpha_2 = 1, \cdots, A - A_1 - A_2. \tag{41}$$

Besides, all the previously established properties (22) - (24) are kept.

Thus at this stage, A_2 one-linked chains of second-class constraints are obtained. In addition, the constraints of different chains are in involution on Σ with each other and with all other constraints.

Next one must consider the constraints $\phi_{\alpha_2}^{m_{\alpha_2}}$, $\alpha_2 = 1, \cdots, A - A_1 - A_2$, and the matrix $\left\| \{\phi_{\alpha_2}^1, \phi_{\beta_2}^3\} \right\|$. With the help of the Jacobi identity we obtain

$$\{\phi_{\alpha_2}^1, \phi_{\beta_2}^3\} \overset{\Sigma}{=} -\{\phi_{\beta_2}^1, \phi_{\alpha_2}^3\}. \tag{42}$$

Let

$$\text{rank} \left\| \{\phi_{\alpha_2}^1, \phi_{\beta_2}^3\} \right\| \overset{\Sigma}{=} A_3 = 2R_3 < A - A_1 - A_2. \tag{43}$$

We shall reckon the principal minor of rank A_3, disposed in the left upper corner of this matrix, to be not equal to zero. We consider that

$$\mathbf{K}^{13} = \left\| \{\phi_{a_3}^1, \phi_{b_3}^3\} \right\|, \qquad a_3, b_3 = 1, \cdots, A_3.$$

We have $F_{11}^{13} = 0$. Renumbering these constraints we shall attain that $F_{12}^{13} \neq 0$. We make the transformation

$$\begin{array}{ll} {}^1\phi_1^1 = \phi_1^1, & {}^1\phi_2^1 = \phi_2^1, \\ {}^1\phi_a^1 = \phi_a^1 + {}^1u_{a1}\,\phi_1^1 + {}^1u_{a2}\,\phi_2^1, & a = 3, \cdots, A_3 \end{array} \tag{44}$$

and, hence,

$$\begin{array}{ll} {}^1\phi_1^3 = \phi_1^3, & {}^1\phi_2^3 = \phi_2^3, \\ {}^1\phi_a^3 = \phi_a^3 + {}^1u_{a1}\,\phi_1^3 + {}^1u_{a2}\,\phi_2^3, & a = 3, \cdots, A_3. \end{array} \tag{45}$$

Coefficients ${}^1u_{a1}$ and ${}^1u_{a2}$ are taken as

$$^1u_{a1} = \frac{D_{2a}^{13}}{D_{12}^{13}}, \qquad ^1u_{a2} = -\frac{D_{1a}^{13}}{D_{12}^{13}} \tag{46}$$

to satisfy the requirement

$$\{ {}^1\phi_1^1, \, {}^1\phi_a^3 \} \overset{\Sigma}{=} 0, \quad \{ {}^1\phi_2^1, \, {}^1\phi_a^3 \} \overset{\Sigma}{=} 0, \qquad a = 3, \cdots, A_3.$$

¿From here

$$\{ {}^1\phi_a^1, \, {}^1\phi_1^3 \} \overset{\Sigma}{=} 0, \quad \{ {}^1\phi_a^1, \, {}^1\phi_2^3 \} \overset{\Sigma}{=} 0, \qquad a = 3, \cdots, A_3.$$

Thus we have

$$^1\mathbf{K}^{13} \overset{\Sigma}{=} \left(\begin{array}{cc|c} 0 & F_{12}^{13} & \\ -F_{12}^{13} & 0 & \mathbf{O} \\ \hline & \mathbf{O} & \left\| {}^1\mathcal{F}_{ab} \right\| (a, b = 3, 4, \cdots, A_3) \end{array} \right).$$

By continuing this process the matrix $\mathbf{K}^{13}|_\Sigma$ will be represented in the quasidiagonal form with only nonvanishing elements along the principal diagonal $F^{1\,3}_{a_3 b_3} \neq 0$ (where if $a_3 = 2k+1$, $b_3 = 2k+2$ and conversely; $k = 0, 1, \cdots, A_3 - 2$).

Again we have the relations:

$$\{\psi^2_{a_3}, \psi^2_{b_3}\} \stackrel{\Sigma}{=} -\{\psi^1_{a_3}, \psi^3_{b_3}\}, \tag{47}$$

$$\{\psi^2_{a_3}, \psi^3_{b_3}\} \stackrel{\Sigma}{=} 0, \quad \{\psi^3_{a_3}, \psi^3_{b_3}\} \stackrel{\Sigma}{=} 0. \tag{48}$$

Thus, two-linked doubled chains of second-class constraints are obtained. Constraints of such different formations are in involution on Σ with each other and with all other constraints, since all the previously established properties are kept. Besides one can receive

$$\{\psi^{m_{a_3}}_{a_3}, \phi^{m_{\alpha_3}}_{\alpha_3}\} \stackrel{\Sigma}{=} 0, \qquad m_{a_3} = 1, 2, 3, \quad \alpha_3 = 1, \cdots, A - \sum_1^3 A_j \tag{49}$$

making the equivalence transformation

$$^1\phi^{m_{\alpha_3}}_{\alpha_3} = \phi^{m_{\alpha_3}}_{\alpha_3} + C^{m_{\alpha_3} m_{a_3}}_{\alpha_3 \; a_3} \; \psi^{m_{a_3}}_{a_3}. \tag{50}$$

Turning to the remaining constraints $\phi^{m_{\alpha_3}}_{\alpha_3}$ one must iterate the above procedure n times. Besides at every i-th stage we consider the constraint set $\phi^{m_{\alpha_{i-1}}}_{\alpha_{i-1}}$ ($\alpha_{i-1} = 1, \cdots, A - \sum_1^i A_j$) with which the constraints chains, already exhibiting their nature of second class at $i-1$ previous stages of our procedure, are in involution on Σ and suppose that

$$\mathrm{rank} \left\| \{\phi^1_{\alpha_{i-1}}, \phi^i_{\beta_{i-1}}\} \right\| \stackrel{\Sigma}{=} A_i < A - \sum_1^i A_j. \tag{51}$$

Further we have the relation [7]:

$$\{\phi^1_{\alpha_{i-1}}, \phi^i_{\beta_{i-1}}\} \stackrel{\Sigma}{=} (-1)^i \{\phi^1_{\beta_{i-1}}, \phi^i_{\alpha_{i-1}}\}. \tag{52}$$

Renumbering the constraints we obtain that the principal minor in the left upper corner of the matrix $\left\| \{\phi^1_{\alpha_{i-1}}, \phi^i_{\beta_{i-1}}\} \right\|$ have the rank A_i. Considering it

$$\mathbf{K}^{1i} = \left\| \{\phi^1_{a_i}, \phi^i_{b_i}\} \right\|, \quad a_i, b_i = 1, \cdots, A_i$$

we see that the matrix $\mathbf{K}^{1i}$ is (anti)symmetric for (odd) even i on Σ (furthermore, its rank is even for odd i). After the (quasi)diagonalization of $\mathbf{K}^{1i}|_\Sigma$ its only nonvanishing elements are (for odd i, $F^{1\,i}_{a_i b_i} \neq 0$, where if $a_i = 2k+1$, $b_i = 2k+2$ and conversely; $k = 0, 1, \cdots, A_i - 2$), for even i, $F^{1\,i}_{a_i a_i} \neq 0$, $a_i = 1, \cdots, A_i$.

Furthermore, with the help of the Jacobi identity we have [7]

$$\{\psi^{i-l}_{a_i}, \psi^{l+1}_{b_i}\} \stackrel{\Sigma}{=} (-1)^l \{\psi^1_{a_i}, \psi^i_{b_i}\}, \quad l = 0, 1, \cdots, i-1, \tag{53}$$

$$\{\psi^j_{a_i}, \psi^k_{b_i}\} \stackrel{\Sigma}{=} 0, \quad j + k \neq i + 1. \tag{54}$$

And also, making the transformation

$$^1\phi^{m_{\alpha_i}}_{\alpha_i} = \phi^{m_{\alpha_i}}_{\alpha_i} + C^{m_{\alpha_i} m_{a_i}}_{\alpha_i \; a_i} \; \psi^{m_{a_i}}_{a_i}, \quad m_{a_i} = 1, \cdots, i, \quad \alpha_i = 1, \cdots, A - \sum_1^i A_j, \tag{55}$$

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one may obtain
$$\{\psi_{a_i}^{m_{a_i}}, \phi_{\alpha_i}^{m_{\alpha_i}}\} \overset{\Sigma}{=} 0. \tag{56}$$

Thus, at the i-th stage we determine A_i $i-1$-linked chains of second-class constraints (doubled for odd i) which are in involution on Σ with remaining constraints, since all the previously established properties are kept also.

If after carrying out certain n-th stage it is found that

$$\text{rank} \left\| \{\phi_{\alpha_n}^{m_{\alpha_n}}, \phi_{\beta_n}^{m_{\beta_n}}\} \right\| \overset{\Sigma}{=} 0, \qquad \alpha_n, \beta_n = 1, \cdots, A - \sum_1^n A_j, \tag{57}$$

then these remaining constraints $\phi_{\alpha_n}^{m_{\alpha_n}}$ are all of first class.

So, the final set of constraints (Φ, Ψ) is obtained from the initial one $\phi_\alpha^{m_\alpha}$ by the equivalence transformation

$$\begin{pmatrix} \boldsymbol{\Phi} \\ \boldsymbol{\Psi} \end{pmatrix} = \prod_{a=1}^X \mathbf{S}^a \begin{pmatrix} \phi \end{pmatrix}, \qquad \det \prod_{a=1}^X \mathbf{S}^a \overset{\Sigma}{\neq} 0 \tag{58}$$

where X is equal to the number of all accomplished stages , $\mathbf{S}^a$ is the matrix of the equivalence transformation of each stage.

The total Hamiltonian assumes the final form

$$H_T = H + u_\alpha \Phi_\alpha^1, \tag{59}$$

where

$$H = H_c + \sum_{i=1}^n (\mathbf{K}^{1\ i})_{b_i\ a_i}^{-1} \{\Psi_{a_i}^i, H_c\} \Psi_{b_i}^1$$

is a first-class function [1], H_c is the canonical Hamiltonian, u_α are the Lagrange multipliers.

Thus, in the Dirac approach, we succeeded in obtaining the canonical set of constraints with properties analogous to the ones in ref. [7] without terms quadratic in constraints in the final form of the total Hamiltonian.

3   Conclusion

In the framework of the original generalized Hamiltonian formalism [1] (without modifications) we have developed a separation scheme of constraints into the first- and second-class ones on the basis of passing to an equivalent canonical set of constraints and have determined the general structure of second-class constraints which is in accordance with the one in the approaches [6, 7]. The latter has permitted us to use the classification of constraints and terminology of paper [7]. That the maximal partition of the set of constraints is achieved and the canonical set of constraints is obtained, is seen from that each second-class constraint of the final set has the vanishing (on the constraint surface) Poisson brackets with all the constraints of the system except one, and the first-class constraints have the vanishing Poisson brackets with all the constraints. These precisely properties will be needed us in following paper II at deriving local-symmetry transformations.

The important feature of our procedure is that each subsequent stage preserves the properties of transformed constraints, which were obtained at preceding stage. This allowed to separate at each stage the second-class constraints. Note that in the generalized Hamiltonian approach there exists a clear distinction between primary constraints, which have pure kinematic charakter as arising only from the definitions of momenta, and the constraints of subsequent stages of the Dirac scheme for breeding the constraints, which uses the equations of motion. It was important also (for following derivation of local-symmetry transformations) to preserve this distinction in the final set of constraints. Therefore our procedure is constructed so that the secondary, tertiary, etc. constraints of canonical set do not mix themselves into primary constraints.

Note that the procedure, proposed in the paper, does not broach questions of separating functionally-independent constraints satisfying the regularity conditions. Discussion of these questions can be found in ref.[4].

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